Math 255A Lecture 7 Notes

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1 Locally Convex Spaces

1.1 Topologies induced by seminorms

Let V be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, and let $(p_\alpha)_{\alpha \in A}$ be a family of seminorms on V. We may introduce a topology on V as follows:

$O \subseteq V$ is open if for any $x \in O$, there exists $\varepsilon > 0$ and finitely many seminorms $p_{\alpha_1}, \dots, p_{\alpha_J}$ such that $N_{\alpha_1,\dots,\alpha_J,\varepsilon} = \bigcap_{j=1}^J \{y \in V : p_{\alpha_j}(y-x) < \varepsilon\} \subseteq O$. This defines a topology on V, and the sets $N_{\alpha_1,\dots,\alpha_J,\varepsilon}$ are open. We shall assume that $(p_\alpha)_{\alpha \in A}$ separates points: $p_\alpha(x) = 0$ for all α iff $x = 0$.

Remark 1.1. An open neighborhood of 0 of the form $\bigcap_{j=1}^J \{x \in V : p_{\alpha_j}(x) < \varepsilon\}$ is balanced and convex.

Definition 1.1. A vector space with a topology defined by a family of seminorms is called a **locally convex space**.

Proposition 1.1. *A locally convex space is Hausdorff.*

Proof. Let $x, y \in V$ be distinct, and let $\alpha \in A$ be such that $p_\alpha(x-y) \neq 0$. Then the open sets $O_x = \{z \in V : p_\alpha(x-z) < p_\alpha(x-y)/4\}$ and $O_x = \{z \in V : p_\alpha(y-z) < p_\alpha(x-y)/4\}$ are disjoint. $\square$

Remark 1.2. In a locally convex space V, the vector operations $+ : V \times V \to V$ and $\cdot : K \times V \to V$, given by $(x,y) \mapsto x+y$ and $(a,x) \mapsto ax$ respectively, are continuous. In particular, translations $x \mapsto x+y$ are homeomorphisms.

Remark 1.3. In a locally convex space, the convex, balanced, open sets of the form $\bigcap_{j=1}^J \{x \in V : p_{\alpha_j}(x) < \varepsilon\}$ form a fundamental system of neighborhoods of 0. Conversely, assume that V is a vector space with a Hausdorff topology in which the vector operations are continuous. Assume that 0 has a fundamental system of neighborhoods which are convex and balanced. Then V is a locally convex space. Let N be such a neighborhood of 0, and let p be gauge of N, $p(x) = \inf\{t > 0 : x/t \in N\}$. Then we know that p is a seminorm on V, and $N = \{x \in V : p(x) < 1\}$.

1.2 Continuity of seminorms

Proposition 1.2. *Let V be a locally convex space with the topology defined by $(p_\alpha)_{\alpha \in A}$. A seminorm p on V is continuous if and only if there is some constant $C > 0$ and $\alpha_1, \dots, \alpha_J$ such that $p(x) \leq C \sum_{j=1}^J p_{\alpha_j}(x)$ for all $x \in V$.*

Proof. We have $|p(x+y) - p(y)| \leq p(x)$, so p is continuous if and only if p is continuous at 0. To show that the condition for continuity is sufficient, we need that for all $\varepsilon > 0$, there exists a neighborhood of $0 \in V$ such that $x \in U \implies p(x) < \varepsilon$. We can take $p_{\alpha_j}(x) < \delta$ for $1 \leq j \leq J$ and $\delta > 0$.

On the other hand, if p is continuous at 0, then there is a neighborhood U of 0 such that $x \in U \implies p(x) < 1$. Thus, there exist $\varepsilon > 0$ and seminorms $p_{\alpha_1}, \dots, p_{\alpha_J}$ such that $p_{\alpha_j}(x) < \varepsilon \, \forall j \in \{1, \dots, J\} \implies p(x) < 1$. Equivalently, if $t > 0$ and x is replaced by tx, $tp_{\alpha_j}(x) < \varepsilon \implies tp(x) < 1$. Take

$$t = \frac{\varepsilon}{\sum_{j=1}^J p_{\alpha_j}(x) + \mu},$$

where $\mu > 0$. We get

$$p(x) < \frac{1}{\varepsilon} \left(\sum_{j=1}^J p_{\alpha_j}(x) + \mu \right)$$

for all $\mu > 0$. $\square$

Remark 1.4. Assume that we have 2 systems of seminorms on V, $(p_\alpha)_{\alpha \in A}$ and $(q_\beta)_{\beta \in B}$. The locally complex topology defined by (p_α) is stronger (has more open sets) the locally convex topology generated by (q_β) if and only if for any $\beta \in B$, we have $q_\beta(x) \leq C \sum_{j=1}^J p_{\alpha_j}(x)$ for all $x \in V$.

Example 1.1. The space $V = C(\mathbb{R})$ becomes a locally convex space with the topology defined by the seminorms $p_n(f) = \sup_{|x| \leq n} |f(x)|$. This topology cannot be defined by a single seminorm p. Otherwise, we would have that for every n, there is a constant $C_n > 0$ such that $p_n(f) \leq C_n p(f)$ for every $f \in C(\mathbb{R})$. We can choose $f \in C(\mathbb{R})$ such that $f(n) = nC_n$ for all n, contradicting this inequality when n is large.

Next time, we will show that a locally convex topology is metrizable if and only if it can be defined by countable many seminorms.